Generating Functions

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$$\begin{cases} \langle g_n \rangle \to G(z) = \sum_{n\geq 0} g_n z^n = g_0 + g_1 z + \cdots + g_n z^n + \cdots \\ \langle f_n \rangle \to F(z) = \sum_{n\geq 0} f_n z^n = f_0 + f_1 z + \cdots + f_n z^n + \cdots \end{cases}$$

$$g_n = [z^n]\, G(z)$$

Table 334 Generating function manipulations.

$$\alpha F(z) + \beta G(z) = \sum_n (\alpha f_n + \beta g_n) z^n$$

$$z^m G(z) = \sum_n g_{n-m} z^n, \quad (\text{integer } m \geq 0) = g_0 z^m + g_1 z^{m+1} + \cdots$$

$$\frac{G(z) - g_0 - g_1 z - \cdots - g_{m-1} z^{m-1}}{z^m} = \sum_{n\geq 0} g_{n+m} z^n, \quad (\text{integer } m \geq 0) = g_m + g_{m+1} z + \cdots$$

$$G(cz) = \sum_n c^n g_n z^n$$

$$G'(z) = \sum_n (n+1) g_{n+1} z^n$$

$$zG'(z) = \sum_n n g_n z^n$$

$$\int_0^z G(t)\,dt = \sum_{n\geq 1} \frac{1}{n} g_{n-1} z^n$$

$$F(z)G(z) = \sum_n \Big(\sum_k f_k g_{n-k}\Big) z^n$$

$$f_0 g_n + f_1 g_{n-1} + \cdots + f_n g_0 = \sum_{\substack{k+h=n \\ h,k\geq 0}} f_k g_h$$

$$(1 + z + z^2 + \cdots + z^n + \cdots) G(z) = \frac{1}{1-z} G(z) = \sum_n \Big(\sum_{k\leq n} g_k\Big) z^n$$

- $$\begin{cases} \dfrac{G(z)+G(-z)}{2} = g_0 + g_2 z^2 + g_4 z^4 + \cdots \\ \dfrac{G(z)-G(-z)}{2} = g_1 z + g_3 z^3 + g_5 z^5 + \cdots \end{cases}$$

- $$\sum_n F_{2n} z^{2n} = \frac{1}{2}\left(\frac{z}{1-z-z^2} + \frac{-z}{1+z-z^2}\right) = \frac{z^2}{1-3z^2+z^4}$$

$$\sum_n F_{2n} z^n = \frac{z}{1-3z+z^2}$$

Table 335 Simple sequences and their generating functions.

sequence	generating function	closed form
$\langle 1,0,0,0,0,0,\ldots\rangle$	$\sum_{n\geq 0}[n=0]\,z^n$	1
$\langle 0,\ldots,0,1,0,0,\ldots\rangle$	$\sum_{n\geq 0}[n=m]\,z^n$	z^m
$\langle 1,1,1,1,1,1,\ldots\rangle$	$\sum_{n\geq 0} z^n$	$\dfrac{1}{1-z}$
$\langle 1,-1,1,-1,1,-1,\ldots\rangle$	$\sum_{n\geq 0}(-1)^n z^n$	$\dfrac{1}{1+z}$
$\langle 1,0,1,0,1,0,\ldots\rangle$	$\sum_{n\geq 0}[2\backslash n]\,z^n$	$\dfrac{1}{1-z^2}$
$\langle 1,0,\ldots,0,1,0,\ldots,0,1,0,\ldots\rangle$	$\sum_{n\geq 0}[m\backslash n]\,z^n$	$\dfrac{1}{1-z^m}$
$\langle 1,2,3,4,5,6,\ldots\rangle$	$\sum_{n\geq 0}(n+1)\,z^n$	$\dfrac{1}{(1-z)^2}$
$\langle 1,2,4,8,16,32,\ldots\rangle$	$\sum_{n\geq 0} 2^n z^n$	$\dfrac{1}{1-2z}$
$\langle 1,4,6,4,1,0,0,\ldots\rangle$	$\sum_{n\geq 0}\binom{4}{n} z^n$	$(1+z)^4$
$\langle 1,c,\binom{c}{2},\binom{c}{3},\ldots\rangle$	$\sum_{n\geq 0}\binom{c}{n} z^n$	$(1+z)^c$
$\left\langle 1,c,\binom{c+1}{2},\binom{c+2}{3},\ldots\right\rangle$	$\sum_{n\geq 0}\binom{c+n-1}{n} z^n$	$\dfrac{1}{(1-z)^c}$
$\langle 1,c,c^2,c^3,\ldots\rangle$	$\sum_{n\geq 0} c^n z^n$	$\dfrac{1}{1-cz}$
$\left\langle 1,\binom{m+1}{m},\binom{m+2}{m},\binom{m+3}{m},\ldots\right\rangle$	$\sum_{n\geq 0}\binom{m+n}{m} z^n$	$\dfrac{1}{(1-z)^{m+1}}$
$\langle 0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\rangle$	$\sum_{n\geq 1}\frac{1}{n} z^n$	$\ln\dfrac{1}{1-z}$
$\langle 0,1,-\frac{1}{2},\frac{1}{3},-\frac{1}{4},\ldots\rangle$	$\sum_{n\geq 1}\frac{(-1)^{n+1}}{n} z^n$	$\ln(1+z)$
$\langle 1,1,\frac{1}{2},\frac{1}{6},\frac{1}{24},\frac{1}{120},\ldots\rangle$	$\sum_{n\geq 0}\frac{1}{n!} z^n$	e^z

- $$\frac{1}{(1-z)^{m+1}} = \sum_{n\geq 0}\binom{-m-1}{n}(-z)^n = \sum_{n\geq 0}\frac{(-m-1)\cdots(-m-n)}{n!}(-1)^n z^n = \sum_{n\geq 0}\binom{m+n}{n} z^n = \binom{m}{m}+\binom{m+1}{m}z+\binom{m+2}{m}z^2+\cdots$$

$\frac{d}{dm}\downarrow$

- $$\frac{1}{(1-z)^{m+1}}\ln\frac{1}{1-z} = \sum_{n\geq 0}\frac{(m+n)\cdots(m+1)}{n!}\left[\frac{1}{m+n}+\cdots+\frac{1}{m+1}\right]z^n = \sum_{n\geq 0}(H_{m+n}-H_m)\binom{m+n}{m}z^n$$

- $$T_{m,n} = \sum_{0\leq k<n}\binom{k}{m}\frac{1}{n-k} = [z^n]\left(\binom{0}{m}+\binom{1}{m}z+\cdots+\binom{n-1}{m}z^{n-1}+\cdots\right)\left(\frac{z}{1}+\frac{z^2}{2}+\cdots+\frac{z^n}{n}+\cdots\right)$$

(Ex. 5-58)
$$= [z^n]\,\frac{z^m}{(1-z)^{m+1}}\ln\frac{1}{1-z} \overset{n\leftarrow n-m}{=\!=} (H_n-H_m)\binom{n}{m}$$

$(m=0)$

$$\frac{1}{1-z} \ln \frac{1}{1-z} = \sum_{n\ge 0} H_n z^n$$

Table 351 Generating functions for special numbers.

$$\frac{1}{(1-z)^{m+1}} \ln \frac{1}{1-z} = \sum_{n\geqslant 0} (H_{m+n} - H_m) \binom{m+n}{n} z^n \qquad (7.43)$$

$$\frac{z}{e^z - 1} = \sum_{n\geqslant 0} B_n \frac{z^n}{n!} \qquad (7.44)$$

$$\frac{F_m z}{1 - (F_{m-1} + F_{m+1})z + (-1)^m z^2} = \sum_{n\geqslant 0} F_{mn} z^n \qquad (7.45)$$

$$\sum_k \left\{ {m \atop k} \right\} \frac{k!\, z^k}{(1-z)^{k+1}} = \sum_{n\geqslant 0} n^m z^n \qquad (7.46)$$

$$(z^{-1})^{\overline{-m}} = \frac{z^m}{(1-z)(1-2z)\ldots(1-mz)} = \sum_{n\geqslant 0} \left\{ {n \atop m} \right\} z^n \qquad (7.47)$$

$$z^{\overline{m}} = z(z+1)\ldots(z+m-1) = \sum_{n\geqslant 0} \left[{m \atop n} \right] z^n \qquad (7.48)$$

$$(e^z - 1)^m = m! \sum_{n\geqslant 0} \left\{ {n \atop m} \right\} \frac{z^n}{n!} \qquad (7.49)$$

$$\left(\ln \frac{1}{1-z} \right)^m = m! \sum_{n\geqslant 0} \left[{n \atop m} \right] \frac{z^n}{n!} \qquad (7.50)$$

$$\left(\frac{z}{\ln(1+z)} \right)^m = \sum_{n\geqslant 0} \frac{z^n}{n!} \left\{ {m \atop m-n} \right\} \Big/ \binom{m-1}{n} \qquad (7.51)$$

$$\left(\frac{z}{1-e^{-z}} \right)^m = \sum_{n\geqslant 0} \frac{z^n}{n!} \left[{m \atop m-n} \right] \Big/ \binom{m-1}{n} \qquad (7.52)$$

$$e^{z+wz} = \sum_{m,n\geqslant 0} \binom{n}{m} w^m \frac{z^n}{n!} \qquad (7.53)$$

$$e^{w(e^z-1)} = \sum_{m,n\geqslant 0} \left\{ {n \atop m} \right\} w^m \frac{z^n}{n!} \qquad (7.54)$$

$$\frac{1}{(1-z)^w} = \sum_{m,n\geqslant 0} \left[{n \atop m} \right] w^m \frac{z^n}{n!} \qquad (7.55)$$

$$\frac{1-w}{e^{(w-1)z} - w} = \sum_{m,n\geqslant 0} \left\langle {n \atop m} \right\rangle w^m \frac{z^n}{n!} \qquad (7.56)$$

Solve problems (1) recurrence of g_n + δ term
(〃 g_n) (2) $G(z)$ equation
(3) solve $G(z)$
(4) expand $G(z)$, get g_n

• **Example 1** (Fibonacci)

$$\begin{cases} g_0 = 0; \quad g_1 = 1; \\ g_n = g_{n-1} + g_{n-2}, \quad \text{for } n \geq 2. \end{cases}$$

n	$\cdots$ -3	-2	-1	0	1	2	3	4	5
g_n	$\cdots$ 0	0	0	0	1	1	2	3	5

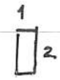
解 $g_n = g_{n-1} + g_{n-2} + \delta_{n=1}, \quad \forall n$

$$\sum_n g_n z^n = \sum_n g_{n-1} z^n + \sum_n g_{n-2} z^n + \sum_n \delta_{n=1} z^n$$

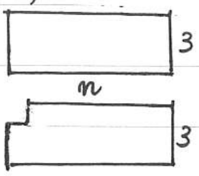
$$G(z) = \sum_n g_n z^{n+1} + \sum_n g_n z^{n+2} + z$$

$$= z\,G(z) + z^2 G(z) + z$$

• **Example 3** (Mutual recurrences) with $\square^{1}_{2}$

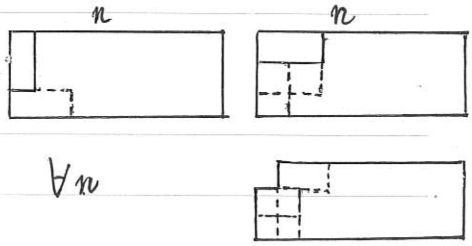


$$\begin{cases} U_n = \text{\# ways to pave } \boxed{\quad n \quad}\,3 = ? \\ V_n = \quad 〃 \quad 〃 \end{cases}$$



n	0	1	2	3	4	5	6	7
U_n	1	0	3	0	11	0	41	0
V_n	0	1	0	4	0	15	0	56

解 $$\begin{cases} U_0 = 1, \quad U_1 = 0; \quad V_0 = 0, \quad V_1 = 1; \\ U_n = 2V_{n-1} + U_{n-2}, \quad V_n = U_{n-1} + V_{n-2}, \quad (n \geq 2) \end{cases}$$



$\Rightarrow$ $U_n = 2V_{n-1} + U_{n-2} + [n=0], \quad V_n = U_{n-1} + V_{n-2}, \quad \forall n$

$$U(z) = 2zV(z) + z^2 U(z) + 1, \quad V(z) = zU(z) + z^2 V(z)$$

$$U(z) = \frac{1 - z^2}{1 - 4z^2 + z^4}; \quad V(z) = \frac{z}{1 - 4z^2 + z^4}.$$

$\begin{cases} U: \text{偶函數} \\ V: \text{奇 〃} \end{cases}$

$$\Rightarrow \begin{cases} V_{2n+1} = W_n = \frac{3+2\sqrt{3}}{6}(2+\sqrt{3})^n + \frac{3-2\sqrt{3}}{6}(2-\sqrt{3})^n; \\ U_{2n} = W_n - W_{n-1} = \frac{3+\sqrt{3}}{6}(2+\sqrt{3})^n + \frac{3-\sqrt{3}}{6}(2-\sqrt{3})^n \end{cases}$$

其中 $\sum_{n \geq 0} W_n z^n = \frac{1}{1-4z+z^2}$

$$= \frac{(2+\sqrt{3})^n}{3-\sqrt{3}} + \frac{(2-\sqrt{3})^n}{3+\sqrt{3}}.$$

P_n	N_n	D_n	Q_n	C_n	= # ways to pay n¢
①	①	①	①	①	
	⑤	⑤	⑤	⑤	
		⑩	⑩	⑩	
			㉕	㉕ + 50	

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Example 4 How many ways to pay 50¢ with ?

$$\begin{cases} P = \cancel{\mathbb{1}} + ① + ①① + ①①① + ①①①① + \cdots \\ \quad = \cancel{\mathbb{1}} + ① + ①^2 + ①^3 + ①^4 + \cdots. \\ N = (\cancel{\mathbb{1}} + ⑤ + ⑤^2 + ⑤^3 + ⑤^4 + \cdots)P, \\ D = (\cancel{\mathbb{1}} + ⑩ + ⑩^2 + ⑩^3 + ⑩^4 + \cdots)N, \\ Q = (\cancel{\mathbb{1}} + ㉕ + ㉕^2 + ㉕^3 + ㉕^4 + \cdots)D; \\ C = (\cancel{\mathbb{1}} + (50) + (50)^2 + (50)^3 + (50)^4 + \cdots)Q. \end{cases}$$

$$\begin{cases} P = 1 + z + z^2 + z^3 + z^4 + \cdots, & = 1/(1-z), \\ N = (1 + z^5 + z^{10} + z^{15} + z^{20} + \cdots)P, & = P/(1-z^5), \\ D = (1 + z^{10} + z^{20} + z^{30} + z^{40} + \cdots)N, & = N/(1-z^{10}) \\ Q = (1 + z^{25} + z^{50} + z^{75} + z^{100} + \cdots)D, & = D/(1-z^{25}) \\ C = (1 + z^{50} + z^{100} + z^{150} + z^{200} + \cdots)Q & = Q/(1-z^{50}) \end{cases}$$

$$\begin{cases} (1-z)P = 1, \\ (1-z^5)N = P, \\ (1-z^{10})D = N, \\ (1-z^{25})Q = D, \\ (1-z^{50})C = Q. \end{cases} \qquad \begin{cases} P_n = P_{n-1} + [n=0], \\ N_n = N_{n-5} + P_n, \\ D_n = D_{n-10} + N_n, \\ Q_n = Q_{n-25} + D_n, \\ C_n = C_{n-50} + Q_n. \end{cases}$$

n	0	5	10	15	20	25	30	35	40	45	50
P_n	1	1	1	1	1	1	1	1	1	1	1
N_n	1	2	3	4	5	6	7	8	9	10	11
D_n	1	2	4	6	9	12	16		25		36
Q_n	1					13					49
C_n	1										50

(1) $C_n = ?$ (# 非負整數解: $p+5q+10d+25g+50f = n$) $= \sum_{p,q,d,g,f \ge 0} z^{p+5q+10d+25g+50f}$

$$C(z) = \sum_{n\ge0} C_n z^n = (1+z+z^2+\cdots)(1+z^5+z^{10}+\cdots)(1+z^{10}+z^{20}+\cdots)(1+z^{25}+z^{50}+\cdots)(1+z^{50}+z^{100}+\cdots)$$

$$= \frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})(1-z^{50})}$$

$$= \frac{1+z+z^2+z^3+z^4}{(1-z^5)\ (1-z^5)(1-z^{10})(1-z^{25})(1-z^{50})}$$

$$= (1+z+z^2+z^3+z^4)(a_0+a_1z^5+a_2z^{10}+\cdots+a_kz^{5k}+\cdots)$$

$$= (a_0+a_0z+a_0z^2+a_0z^3+a_0z^4)+(a_1z^5+a_1z^6+\cdots+a_1z^9)$$

$$+\cdots+(a_kz^{5k}+a_kz^{5k+1}+\cdots+a_kz^{5k+4})+\cdots$$

$$\Rightarrow C_{5k} = C_{5k+1} = C_{5k+2} = C_{5k+3} = C_{5k+4} = a_k$$

$$C_n = a_{\lfloor \frac{n}{5} \rfloor}$$

- $G(z) = \dfrac{1}{(1-z)(1-z)(1-z^2)(1-z^5)(1-z^{10})}$
  $= a_0 + a_1z + a_2z^2 + \cdots + a_kz^k + \cdots$

- $(1-z^{10})^{-5} = \sum_{k\ge0} \binom{k+4}{4} z^{10k}$

- $(1+z+\cdots+z^9)^2(1+z^2+\cdots+z^8)(1+z^5)$
  $= 1 + 2z + 4z^2 + 6z^3 + 9z^4 + 13z^5 + 18z^6 + 24z^7$
  $+ 31z^8 + 39z^9 + 45z^{10} + 52z^{11} + 57z^{12} + 63z^{13}$
  $+ 67z^{14} + 69z^{15} + 69z^{16} + 67z^{17} + 63z^{18}$
  $+ 57z^{19} + 52z^{20} + 45z^{21} + 39z^{22} + 31z^{23}$
  $+ 24z^{24} + 18z^{25} + 13z^{26} + 9z^{27} + 6z^{28}$
  $+ 4z^{29} + 2z^{30} + z^{31}.$
  $= A_0 + A_1z + \cdots + A_{31}z^{31}.$

(2) $a_k = ?$

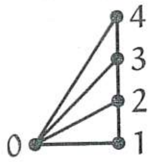
- $G(z) = \sum_{k\ge0} a_k z^k = \dfrac{1}{(1-z)(1-z)(1-z^2)(1-z^5)(1-z^{10})}$

$$= \frac{(1+z+z^2+\cdots+z^9)^2(1+z^2+\cdots+z^8)(1+z^5)}{(1-z^{10})(1-z^{10})\ (1-z^{10})(1-z^{10})(1-z^{10})}$$

$$= (A_0+A_1z+\cdots+A_{30}z^{30}+A_{31}z^{31})\left[\binom{4}{4}+\binom{5}{4}z^{10}+\binom{6}{4}z^{20}+\cdots+\binom{7+4}{4}z^{70}+\binom{8+4}{4}z^{80}+\cdots+\binom{8+4}{4}z^{108}+\cdots\right]$$

- $C_{50} = a_{10} = A_0\binom{5}{4} + A_{10}\binom{4}{4} = 50,$ $\qquad a_{13} = A_3\binom{5}{4} + A_{13}\binom{4}{4}$
- $C_{100} = a_{20} = A_0\binom{6}{4} + A_{10}\binom{5}{4} + A_{20}\binom{4}{4} = 292,$
- $C_{400} = a_{80} = A_0\binom{8+4}{4} + A_{10}\binom{7+4}{4} + A_{20}\binom{6+4}{4} + A_{30}\binom{5+4}{4},$ $\qquad a_{83} = A_3\binom{8+4}{4} + A_{13}\binom{7+4}{4} + A_{23}\binom{6+4}{4}$
- $C_{508} = a_{108} = A_0\binom{8+4}{4} + A_{10}\binom{8+3}{4} + A_{20}\binom{8+2}{4} + A_{30}\binom{8+1}{4},$
- $a_{108+r} = A_r\binom{8+4}{4} + A_{r+10}\binom{8+3}{4} + A_{r+20}\binom{8+2}{4} + A_{r+30}\binom{8+1}{4}$
  $(0 \le r < 10)$

• **Example 6** $f_n = \#$ spanning trees of n-fan.



(n=4)

n	$\cdots -1$	0	1	2	3	4
f_n	$\cdots 0$	0	1	3	8	21

解 (甲) $\begin{cases} f_1 = 1 \\ f_n = f_{n-1} + f_{n-1} + f_{n-2} + \cdots + f_1 + 1 \quad (n \geq 2) \end{cases}$

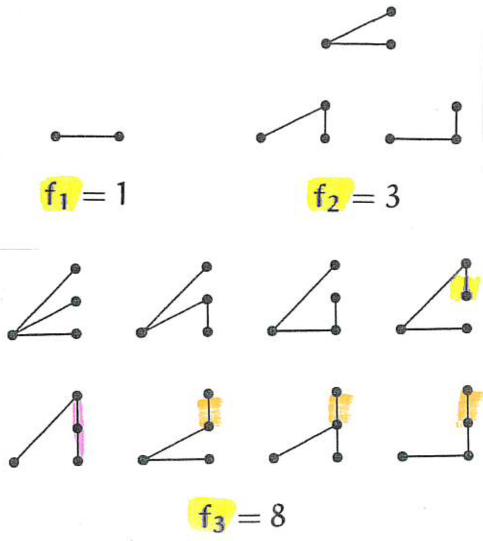
$$\Rightarrow f_n = f_{n-1} + \sum_{k<n} f_k + \delta_{n\geq 1}, \quad \forall n$$

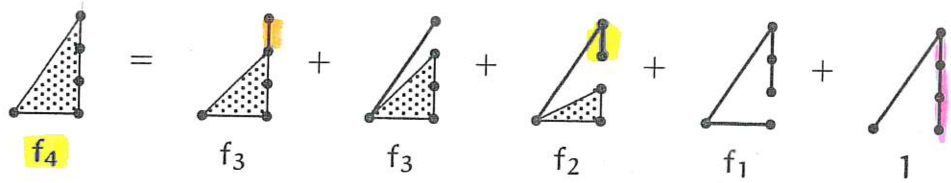
$$F(z) = zF(z) + (z + z^2 + \cdots)F(z) + (z + z^2 + \cdots)$$

$$= zF(z) + \frac{z}{1-z}F(z) + \frac{z}{1-z}$$

$$F(z) = \frac{z}{1-3z+z^2}$$

$$\therefore f_n = F_{2n}$$

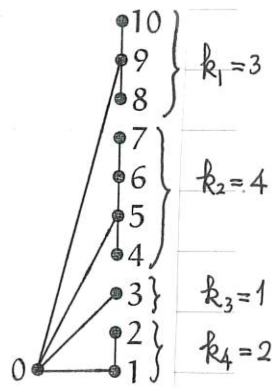




(乙)

$$f_n = \sum_{m>0} \sum_{\substack{k_1+k_2+\cdots+k_m=n \\ k_1,k_2,\ldots,k_m>0}} k_1 k_2 \ldots k_m$$

$$f_4 = 4 + 3\cdot1 + 2\cdot2 + 1\cdot3 + 2\cdot1\cdot1 + 1\cdot2\cdot1 + 1\cdot1\cdot2 + 1\cdot1\cdot1\cdot1 = 21.$$



$$F(z) = \sum_{n>0}\left(\sum_{m>0} \sum_{\substack{k_1+k_2+\cdots+k_m=n \\ k_1,k_2,\ldots,k_m>0}} k_1 k_2 \cdots k_m\right) z^n$$

$$= \sum_{m>0} \sum_{n>0} \sum_{k_1+k_2+\cdots+k_m=n} k_1 k_2 \cdots k_m \, z^{k_1+k_2+\cdots+k_m}$$

$$= \sum_{m>0} \left(\sum_{k_1>0} k_1 z^{k_1}\right)\left(\sum_{k_2>0} k_2 z^{k_2}\right) \cdots \left(\sum_{k_m>0} k_m z^{k_m}\right)$$

$$= \sum_{m>0} G(z)^m \qquad G(z) = \sum_{k>0} k z^k = z + 2z^2 + 3z^3 + \cdots = \frac{z}{(1-z)^2}$$

$$= \frac{G(z)}{1-G(z)}$$

$$= \frac{z}{1-3z+z^2}$$

Catalan numbers

(1) $C_n = \#$ binary trees with n nodes

n	-2	-1	0	1	2	3	4	5	6	7	⋯
C_n	0	0	1	1	2	5	14	42	132	429	⋯

• $C_n = C_{n-1} + C_1 C_{n-2} + C_2 C_{n-3} + \cdots + C_{n-2} C_1 + C_{n-1}$

$= C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0, \ (n \geq 1)$

$= \sum_k C_k C_{n-1-k} + \delta_{n=0}, \quad \forall n$

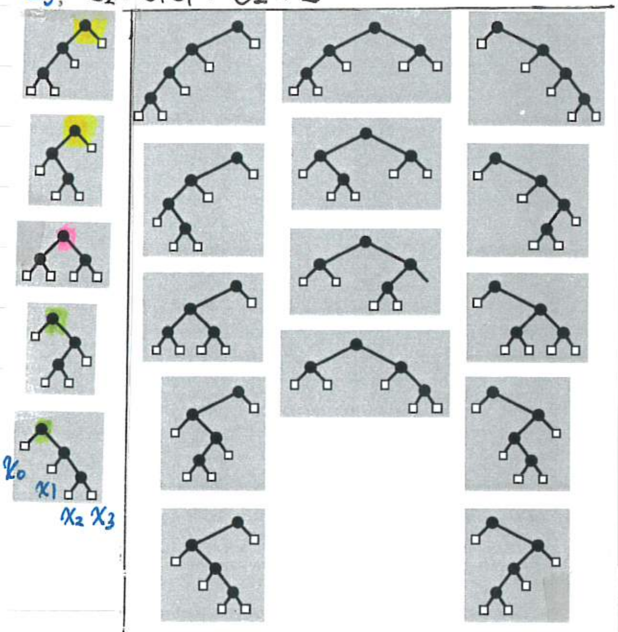
• $C(z) = \sum_n C_n z^n = z\,C(z)^2 + 1$

$C(z) = \dfrac{1 \pm \sqrt{1-4z}}{2z} \overset{\text{取“−”}}{=} \dfrac{1}{2z}\left[1 - \sum_{k \geq 0} \binom{\frac{1}{2}}{k} (-4z)^k\right]$

• $C_n = \dfrac{-1}{2}\binom{\frac{1}{2}}{n+1}(-4)^{n+1} = \dfrac{-1 \cdot \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)\left(-2\right)^{n+1}}{2\,(n+1)!}$

$= \dfrac{1 \cdot 3 \cdots (2n-1)\, 2^n}{(n+1)\, n!} \cdot \dfrac{n!}{n!} = \dfrac{1}{(n+1)}\binom{2n}{n}$

$C_3 = C_2 + C_1 C_1 + C_2 = 5$



$C_4 = C_3 + C_2 C_1 + C_1 C_2 + C_3 = 14$

(2) $C_n = \#$ ways to multiply (parenthesize) $x_0 \cdot x_1 \cdots x_n$

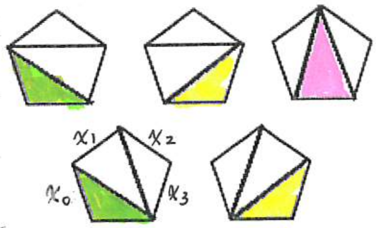
$(x_0 \cdot (x_1 \cdot (x_2 \cdot x_3)))$, $(x_0 \cdot ((x_1 \cdot x_2) \cdot x_3))$, $((x_0 \cdot x_1) \cdot (x_2 \cdot x_3))$,

$C_3 = 5$ $\quad ((x_0 \cdot (x_1 \cdot x_2)) \cdot x_3)$, $(((x_0 \cdot x_1) \cdot x_2) \cdot x_3)$

$= C_2 + C_1 C_1 + C_2$

$C_n = C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1}$

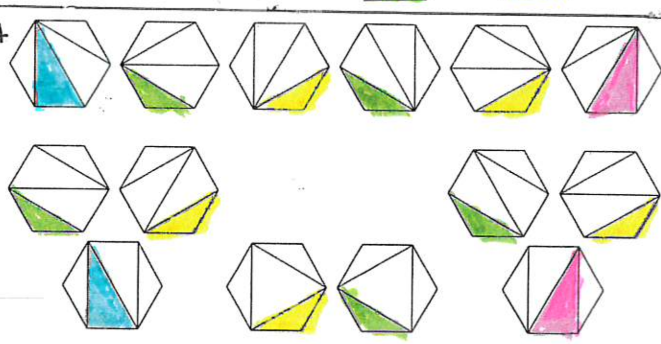
(3) $C_n = \#$ ways to triangulate $(n+2)$-gon $\quad C_3 = 5$,

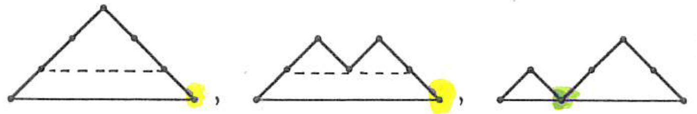


(4) $C_n = \#$ sequences $(a_1, a_2, \ldots, a_{2n})$ $\begin{cases} a_1 + a_2 + \cdots + a_{2n} = 0 \\ a_1 + a_2 + \cdots + a_i \geq 0, & 1 \leq i \leq 2n \end{cases}$

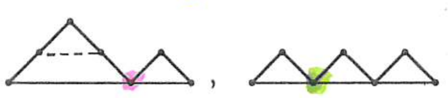
(甲) $a_i = \begin{cases} +1, \ \nearrow, \ (, \ \text{push}, \ 1 \\ -1, \ \searrow, \ ), \ \text{pop}, \ 0 \end{cases}$

$C_4 = 14$





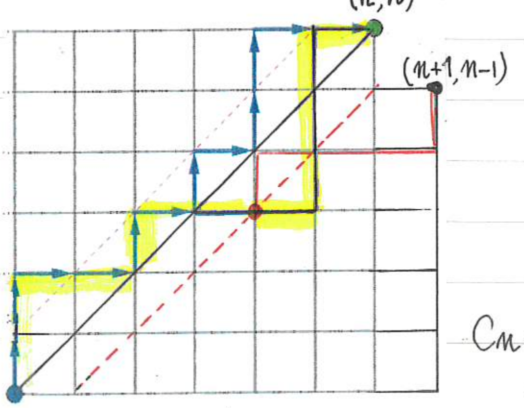
$\begin{cases} (\rightarrow +1 \\ \bullet \rightarrow -1 \end{cases}$



(乙) 1 1 0 0 1 0 0 | 0 1 1 1 0 $\rightarrow (n, n)$ (x)

1 0 0 0 1 $\rightarrow (n+1, n-1)$

$\begin{cases} 1 \text{ 個數} = 3 + \cancel{3}\ 2 = n-1 \\ 0 \quad " \quad = 4 + \cancel{2}\ 3 = n+1 \end{cases}$



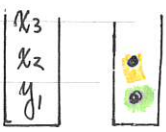
$C_n = \binom{2n}{n} - \binom{2n}{n+1}$

$= \dfrac{1}{n+1}\binom{2n}{n}$

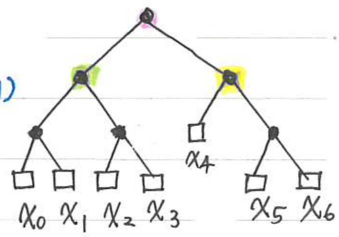
$\boxed{\text{对應}}$ $n=6$

(2) $(((x_0 \cdot x_1) \cdot (x_2 \cdot x_3)) \cdot (x_4 \cdot (x_5 \cdot x_6)))$

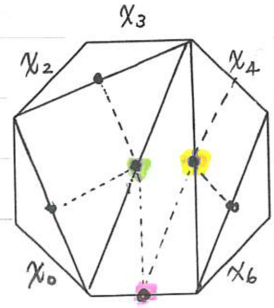
(4) + − + $\quad$ + − − + $\quad$ + + − − −



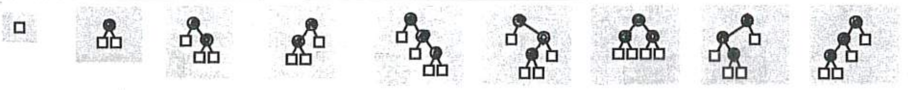
(1)



(3)



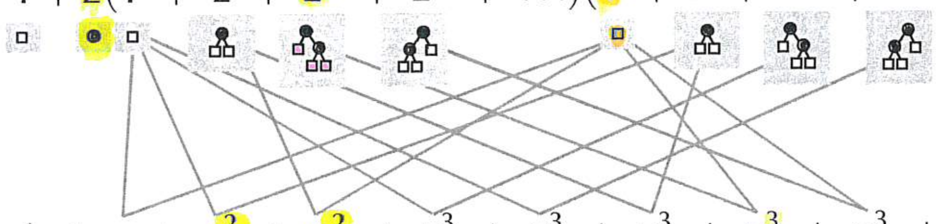
Analytic combinatorics

$\mathcal{T} = \{$  $\}$

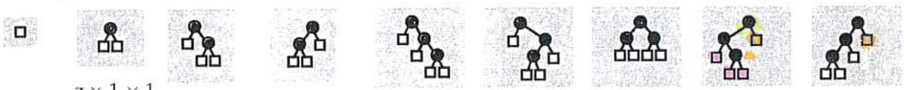
$$T(z) = 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + z^3 + z^3 + \dots$$
$$= 1 + z + 2z^2 + 5z^3 + \dots$$

$$T(z) = 1 + zT(z)^2$$

$$R = 1 + z(1 + z + z^2 + z^2 + \dots)(1 + z + z^2 + z^2 + \dots)$$



$$L = 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + z^3 + z^3 + \dots$$



$z\times1\times1$ $\quad z\times1\times z$ $\quad z\times z\times1$ $\quad z\times1\times z^2$ $\quad z\times1\times z^2$ $\quad z\times z\times z$ $\quad z\times z^2\times1$ $\quad z\times z^2\times1$

- $\mathcal{T} = \{t \mid t:\text{Binary tree}\}$ $\qquad |t| = \begin{cases} \#\text{ nodes of } t, \\ 0, & t = \square \end{cases}$
- $T(z) = \sum_{n\geq0} C_n z^n = \sum_{t\in\mathcal{T}} z^{|t|} = 1 + \sum_{t_L, t_R\in\mathcal{T}} z^{1+|t_L|+|t_R|} = 1 + z\sum_{t_L\in\mathcal{T}} z^{|t_L|}\cdot\sum_{t_R\in\mathcal{T}} z^{|t_R|}$
  $= 1 + zT(z)T(z)$

• Symbolic method

(1) $\mathcal{C}$: class of combinatorial objects
(2) $|w|_{\mathcal{C}}$: size of $w\in\mathcal{C}$, $C_n = \#w$ of size n.
(3) structural relation of $\mathcal{C}$
(4) equation of $C(z) = \sum_{w\in\mathcal{C}} z^{|w|} = \sum_{n\geq0} C_n z^n$

Example: $\mathcal{T}$
$|t| = \#$ nodes of $t\in\mathcal{T}$
$\mathcal{T} \simeq \{\square\} + \mathcal{T}\times\{\bullet\}\times\mathcal{T}$
$T(z) = 1 + T(z)\,z\,T(z)$

• 定理

(1) (disjoint union) $\quad \mathcal{C} = \mathcal{A} + \mathcal{B}$, $\quad |w|_{\mathcal{C}} = \begin{cases} |w|_{\mathcal{A}}, & w\in\mathcal{A} \\ |w|_{\mathcal{B}}, & w\in\mathcal{B}\end{cases}$ $\quad\Rightarrow C(z) = A(z) + B(z)$

(2) (Cartesian product) $\quad \mathcal{C} = \mathcal{A}\times\mathcal{B}$, $\quad |(a,b)|_{\mathcal{C}} = |a|_{\mathcal{A}} + |b|_{\mathcal{B}}$ $\quad\Rightarrow C(z) = A(z)B(z)$

(3) (Sequence) $\quad \mathcal{C} = \mathcal{A}^* = \{\varepsilon\} + \mathcal{A} + \mathcal{A}\times\mathcal{A} + \mathcal{A}\times\mathcal{A}\times\mathcal{A} + \cdots \Rightarrow C(z) = \dfrac{1}{1 - A(z)}$

証: (1) $C(z) = \sum_{w\in\mathcal{C}} z^{|w|} = \sum_{w\in\mathcal{A}} z^{|w|} + \sum_{w\in\mathcal{B}} z^{|w|} = A(z) + B(z)$ $\qquad (C_n = a_n + b_n)$

(2) $C(z) = \sum_{(a,b)\in\mathcal{A}\times\mathcal{B}} z^{|(a,b)|} = \sum_{a\in\mathcal{A}, b\in\mathcal{B}} z^{|a|+|b|} = \sum_{a\in\mathcal{A}} z^{|a|} \sum_{b\in\mathcal{B}} z^{|b|} = A(z)B(z)$ $\quad \left(C_n = \sum_{k=0}^{n} a_k b_{n-k}\right)$

(3) $C(z) = 1 + A(z) + A^2(z) + A^3(z) + \cdots$

• Examples

$0+1 \to z+z$

(1) Binary strings: (甲) $\begin{cases} \mathcal{B} = (0+1)^* \\ |b| = \#\text{ bits in } b, \end{cases}$ $B(z) = \dfrac{1}{1-2z}$ $b_n = 2^n$

(乙) $\mathcal{B} = \varepsilon + (0+1) \times \mathcal{B}$, $B(z) = 1 + 2z B(z)$

(2) Binary strings with no '00': $\mathcal{B}' = \varepsilon + 0 + (1+01) \times \mathcal{B}'$, $B'(z) = \dfrac{1+z}{1-z-z^2}$, $b_n' = F_{n+1} + F_n = F_{n+2}$

$B'(z) = 1 + z + (z+z^2) B'(z)$,

$\begin{cases} b_1' = 2, & 0,\ 1 \\ b_2' = 3, & 01,\ 10,\ 11 \\ b_3' = 5, & 010,\ 011,\ 101,\ 110,\ 111 \end{cases}$

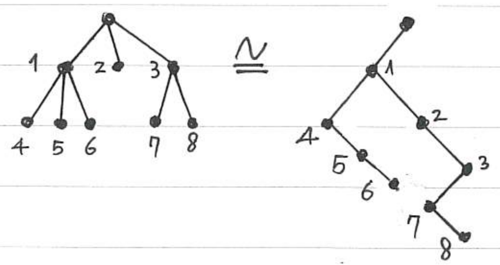
(3) Binary trees: $\begin{cases} \mathcal{T} = \square + \mathcal{T} \times \bullet \times \mathcal{T}, \\ |t| = \#\text{ internal nodes}, \end{cases}$ $T(z) = 1 + z T(z) T(z)$, $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$

(4) Binary trees: $\begin{cases} \mathcal{T} = \square + \mathcal{T} \times \bullet \times \mathcal{T}, \\ |t| = \#\text{ external } '' \end{cases}$ $T^*(z) = z + T^*(z) T^*(z)$, $C_n^* = C_{n-1} = \dfrac{1}{n}\dbinom{2n-2}{n-1}$

$(T^*(z) = z T(z))$

(5) Non-empty General trees: $\mathcal{G} = \{\bullet\} \times \mathcal{G}^*$, $G(z) = \dfrac{z}{1-G(z)}$ $g_n = C_{n-1} = \dfrac{1}{n}\dbinom{2n-2}{n-1}$

$G(z) = z + G(z) G(z)$



(6) Floor tiling: $|t| = \#$ tiles,

$$\begin{aligned} T &= 1 + \square + ◫ + ⊟ + ◫\square + ⊞\square + \square⊟ + \cdots \\ &= 1 + \square(1 + \square + ◫ + ⊟ + \cdots) + ⊟(1 + \square + ◫ + ⊟ + \cdots) \\ &= 1 + \square T + ⊟ T. \end{aligned}$$

$T(z) = 1 + z T(z) + z^2 T(z)$ $T_n = F_{n+1}$

∘ Ordinary G.F.

(Convolution)

$$\begin{cases} A(z) = \sum_{k\geq 0} a_k z^k = a_0 + a_1 z + a_2 z^2 + \cdots + a_6 z^6 + \cdots \\ B(z) = \sum_{k\geq 0} b_k z^k = b_0 + b_1 z + b_2 z^2 + \cdots + b_6 z^6 + \cdots \end{cases}$$

$$A(z)B(z) = C(z) = \sum_{n\geq 0} c_n z^n, \quad c_n = \sum_{0\leq k\leq n} a_k b_{n-k} \qquad (c_6 = a_0 b_6 + a_1 b_5 + a_2 b_4 + \cdots + a_6 b_0)$$

∘ Exponential G.F.

$$\begin{cases} \hat{A}(z) = \sum_{k\geq 0} a_k \frac{z^k}{k!} = a_0 + a_1 \frac{z}{1!} + a_2 \frac{z^2}{2!} + \cdots + a_6 \frac{z^6}{6!} + \cdots \\ \hat{B}(z) = \sum_{k\geq 0} b_k \frac{z^k}{k!} = b_0 + b_1 \frac{z}{1!} + b_2 \frac{z^2}{2!} + \cdots + b_6 \frac{z^6}{6!} + \cdots, \end{cases} \quad \left(\frac{c_6}{6!} = \frac{a_0}{0!}\frac{b_6}{6!} + \frac{a_1}{1!}\frac{b_5}{5!} + \frac{a_2}{2!}\frac{b_4}{4!} + \cdots + \frac{a_6}{6!}\frac{b_0}{0!}\right)$$

$$\hat{A}(z)\hat{B}(z) = \hat{C}(z) = \sum_{n\geq 0} c_n \frac{z^n}{n!}, \quad c_n = \sum_{0\leq k\leq n} \binom{n}{k} a_k b_{n-k}$$

∘ Dirichlet G.F.

$$\begin{cases} \tilde{A}(z) = \sum_{k\geq 1} \frac{a_k}{k^z} = \frac{a_1}{1^z} + \frac{a_2}{2^z} + \frac{a_3}{3^z} + \cdots + \frac{a_6}{6^z} + \cdots \\ \tilde{B}(z) = \sum_{k\geq 1} \frac{b_k}{k^z} = \frac{b_1}{1^z} + \frac{b_2}{2^z} + \frac{b_3}{3^z} + \cdots + \frac{b_6}{6^z} + \cdots \end{cases} \qquad (c_6 = a_1 b_6 + a_2 b_3 + a_3 b_2 + a_6 b_1)$$

$$\tilde{A}(z)\tilde{B}(z) = \tilde{C}(z) = \sum_{m\geq 1} \frac{c_m}{m^z}, \quad c_m = \sum_{d\mid m} a_d b_{\frac{m}{d}} = \sum_{dk=m} a_d b_k \qquad (c = a * b)$$

定理 (1) $\zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots = \frac{1}{1-\frac{1}{2^z}} \cdot \frac{1}{1-\frac{1}{3^z}} \cdot \frac{1}{1-\frac{1}{5^z}} \cdots$ (Rieman Zeta fn)

$$= \left(1 + \frac{1}{2^z} + \frac{1}{2^{2z}} + \frac{1}{2^{3z}} + \cdots\right)\left(1 + \frac{1}{3^z} + \frac{1}{3^{2z}} + \frac{1}{3^{3z}} + \cdots\right)\left(1 + \frac{1}{5^z} + \frac{1}{5^{2z}} + \frac{1}{5^{3z}} + \cdots\right)\left(\cdots\right.$$

(2) $\tilde{\mu}(z) = \frac{\mu(1)}{1^z} + \frac{\mu(2)}{2^z} + \frac{\mu(3)}{3^z} + \frac{\mu(4)}{4^z} + \cdots = \left(1 - \frac{1}{2^z}\right)\left(1 - \frac{1}{3^z}\right)\left(1 - \frac{1}{5^z}\right)\left(\cdots\right.$

(3) $\zeta(z)\tilde{\mu}(z) = 1$

定理 (1) $G(z) = \zeta(z)F(z) \iff F(z) = \tilde{\mu}(z)G(z)$

(2) $g = 1 * f \iff f = \mu * g$

(3) $g_m = \sum_{d\mid m} f_d \iff f_m = \sum_{d\mid m} \mu(d)\, g_{\frac{m}{d}}$ (Möbius inversion)